IV. "On an Instrument for calculating $\left(\int\phi(x)\,\psi(x)\,dx\right)$, the Integral of the Product of two given Functions." By Prof. Sir W. Thomson, LL.D., F.R.S. Received January 28, 1876.

In consequence of the recent meeting of the British Association at Bristol, I resumed an attempt to find an instrument which should supersede the heavy arithmetical labour of calculating the integrals required to analyze a function into its simple harmonic constituents according to the method of Fourier. During many years previously it had appeared to me that the object ought to be accomplished by some simple mechanical means; but it was not until recently that I succeeded in devising an instrument approaching sufficiently to simplicity to promise practically useful results. Having arrived at this stage, I described my proposed machine a few days ago to my brother Professor James Thomson, and he described to me in return a kind of mechanical integrator which had occurred to him many years ago, but of which he had never published any description. I instantly saw that it gave me a much simpler means of attaining my special object than any thing I had been able to think of previously. An account of his integrator is communicated to the Royal Society along with the present paper.

To calculate $\int\phi(x)\,\psi(x)dx$, the rotating disk is to be displaced from a zero or initial position through an angle equal to $\int_0^x\phi(x)\,dx$, while the rolling globe is moved so as always to be at a distance from its zero position equal to $\psi(x)$. This being done, the cylinder obviously turns through an angle equal to $\int_0^x\phi(x)\,\psi(x)\,dx$, and thus solves the problem.

One way of giving the required motions to the rotating disk and rolling globe is as follows:—

On two pieces of paper draw the curves

$$y=\int_0^x\phi(x)\,dx,\text{ and } y=\psi(x).$$

Attach these pieces of paper to the circumference of two circular cylinders, or to different parts of the circumference of one cylinder, with the axis of x in each in the direction perpendicular to the axis of the cylinder. Let the two cylinders (if there are two) be geared together so as that their circumferences shall move with equal velocities. Attached to the framework let there be, close to the circumference of each cylinder, a slide or guide-rod to guide a movable point, moved by the hand of an operator, so as always to touch the curve on the surface of the cylinder, while the two cylinders are moved round.

Two operators will be required, as one operator could not move the two points so as to fulfil this condition—at all events unless the motion were very slow. One of these points, by proper mechanism, gives an

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angular motion to the rotating disk equal to its own linear motion, the other gives a linear motion equal to its own to the centre of the rolling globe.

The machine thus described is immediately applicable to calculate the values H_1, H_2, H_3, &c. of the harmonic constituents of a function $\psi(x)$ in the splendid generalization of Fourier's simple harmonic analysis, which he initiated himself in his solutions for the conduction of heat in the sphere and the cylinder, and which was worked out so ably and beautifully by Poisson*, and by Sturm and Liouville in their memorable papers on this subject published in the first volume of Liouville's 'Journal des Mathématiques.' Thus if

$$\psi(x)=H_1\,\phi_1(x)+H_2\,\phi_2(x)+H_3\,\phi_3(x)+\&c.$$

be the expression for an arbitrary function ψx, in terms of the generalized harmonic functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, &c., these functions being such that

$$\int_0^l \phi_1(x)\,\phi_2(x)\,dx=0,\ \int_0^l \phi_1(x)\,\phi_3(x)\,dx=0,\ \int_0^l \phi_2(x)\,\phi_3(x)=0,\ \&c.,$$

we have

$$H_1=\frac{\int_0^l \phi_1(x)\,\psi(x)\,dx}{\int_0^l \{\phi_1(x)\}^2\,dx},$$

$$H_2=\frac{\int_0^l \phi_2(x)\,\psi(x)\,dx}{\int_0^l \{\phi_2(x)\}^2\,dx},$$

&c.

In the physical applications of this theory the integrals which constitute the denominators of the formulæ for H_1, H_2, &c. are always to be evaluated in finite terms by an extension of Fourier's formula for $\int_0^x x\,u_i^2\,dx$ of his problem of the cylinder † made by Sturm in equation (10), § iv. of his "Mémoire sur une Classe d'Équations à différences partielles" in Liouville's Journal, vol. i. (1836). The integrals in the numerators are calculated with great ease by aid of the machine worked in the manner described above.

The great practical use of this machine will be to perform the simple harmonic Fourier-analysis for tidal, meteorological, and perhaps even astronomical observations. It is the case in which

$$\phi(x)=\frac{\sin}{\cos}(nx);$$

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* His general demonstration of the reality of the roots of transcendental equations essential to this analysis (an exceedingly important step in advance from Fourier's position), which he first gave in the 'Bulletin de la Société Philomathique' for 1828, is reproduced in his 'Théorie Mathématique de la Chaleur,' § 90.

† Fourier's 'Théorie Analytique de la Chaleur,' § 319, page 391 (Paris, 1822).

and the integration is performed through a range equal to $\frac{2i\pi}{n}$ (i any integer) that gives this application. In this case the addition of a simple crank mechanism, to give a simple harmonic angular motion to the rotating disk in the proper period $\frac{2\pi}{n}$, when the cylinder bearing the curve $y=\psi(x)$ moves uniformly, supersedes the necessity for a cylinder with the curve $y=\phi(x)$ traced on it, and an operator keeping a point always on this curve in the manner described above. Thus one operator will be enough to carry on the process; and I believe that in the application of it to the tidal harmonic analysis he will be able in an hour or two to find by aid of the machine any one of the simple harmonic elements of a year's tides recorded in curves in the usual manner by an ordinary tide-gauge—a result which hitherto has required not less than twenty hours of calculation by skilled arithmeticians. I believe this instrument will be of great value also in determining the diurnal, semi-diurnal, ter-diurnal, and quarter-diurnal constituents of the daily variations of temperature, barometric pressure, east and west components of the velocity of the wind, north and south components of the same; also of the three components of the terrestrial magnetic force; also of the electric potential of the air at the point where the stream of water breaks into drops in the atmospheric electrometers, and of other subjects of ordinary meteorological or magnetic observations; also to estimate precisely the variation of terrestrial magnetism in the eleven years sun-spot period, and of sun-spots themselves in this period; also to disprove (or prove, as the case may be) supposed relations between sun-spots and planetary positions and conjunctions; also to investigate lunar influence on the height of the barometer, and on the components of the terrestrial magnetic force, and to find if lunar influence is sensible on any other meteorological phenomena—and if so, to determine precisely its character and amount.

From the description given above it will be seen that the mechanism required for the instrument is exceedingly simple and easy. Its accuracy will depend essentially on the accuracy of the circular cylinder, of the globe, and of the plane of the rotating disk used in it.

For each of the three surfaces a much less elaborate application of the method of scraping than that by which Sir Joseph Whitworth has given a true plane with such marvellous accuracy will no doubt suffice for the practical requirements of the instrument now proposed.